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Stability of Elastic Systems Under Follower Forces

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The stability of a cantilever beam with a tip mass resting on an elastic foundation of the Winkler type and subjected to a follower force at the free end is studied by the finite element method. Instead of the conventional trial-and-error technique, the technique of eigenvalue sensitivity is introduced to rapidly obtain the critical flutter load. The influence of some parameters, such as the modulus of the elastic foundation, the ratio of the cantilever beam mass to the tip mass, and the rotatory inertia of the tip mass, on the critical flutter is investigated. Numerical examples are illustrated to show the rapid convergence rate of the iterative technique proposed in this paper. The excellent agreement of present solutions with the available results is also demonstrated.

I. Introduction

BEAMLIKE structures have been widely used in many industrial fields, such as mechanical, aerospace, and rocket engineering, and, therefore, the problems of vibration and stability of such structural components subjected to nonconservative forces have recently received considerable attention by many researchers. In books by Bolotin[1] and Leipholz,[2] these problems and various types of nonconservative forces often encountered in practical engineering design are well discussed and illustrated. It is shown that these problems have become increasingly important in modern lightweight structure design.

The stability of a cantilever column subjected to a follower force at the free end was first investigated by Beck[3] in 1952. The case of a column with a concentrated mass was presented by Pflüger.[4] After their works, many investigations have been devoted to the determination of the influence quantities on the values of the critical flutter loads of nonconservatively loaded elastic systems. For example, Nemat-Nasser[5] considered the effects of transverse shear deformation and rotatory inertia as well as the internal damping forces corresponding to shear deformation for the Beck's column. Sundararajan[6] took into account the influence of an elastic end support on the stability of Beck's column. Kounadis and Katsikadelis[7-9] generalized Nemat-Nasser's work to assess a variety of parameters, such as elastic spring supports and attached mass, on the critical loads. Anderson[10] presented the influence of a tip mass, internal damping, and an elastic foundation upon the critical flutter load of Beck's column. In these works, analytical methods were used and, therefore, a rather complicated frequency equation belonging to a class of transcendental functions was obtained.

With the advances of computer hardware and software, numerical methods have become popular for analysis of the stability of nonconservatively loaded elastic systems. Of the many numerical methods, the most commonly employed is the finite element method. Barsoum[11] presented the finite element procedure and solution technique to the stability problem of nonconservative systems. Sundararamaiah and Rao[12] used the Timoshenko beam element to determine the critical flutter load of Beck's column. Park[13] studied the dynamic stability of a free-free Timoshenko beam driven by a follower force with controlled directions. Chen and Yang[14] carried out the critical loads of a bimodulus Timoshenko beam subjected to a combined action of a dead load and a follower force. The advantages of systematic formulation and high successful results through the use of the finite element method are clearly shown in those studies.

As indicated in Ref. 11, the critical load determination of nonconservative systems could only be accomplished by a trial method and any acceleration technique, such as the Newton-Raphson method, would not work. In general, the process of the trial-and-error technique can be described briefly as follows. The starting trial load is arbitrarily chosen well below the critical load; the load is then increased in a large increment. At each load increment, the two smallest eigenvalues are found and compared. As soon as the instability region is reached, the load is decreased by one step increment and the system is set back into the stable region and then a much smaller load increment is used until the instability region is again reached. The process is repeated until the desired accuracy is obtained. It can be seen that this process is quite an arduous task and consumes quite a bit of computer time if a sufficient accuracy is required. It is of interest to improve the arduously numerical work just mentioned. To this end, Pedersen and Seyranian[15] took advantage of the technique of sensitivity analysis for problems of dynamic stability, where discretized as well as nondiscretized examples are presented in detail. It is shown that, with the incorporation of the solution to the adjoint problem, a sensitivity analysis of nonconservative problems only needs comparatively few calculations.

The objective of this paper is to extend the basic idea in Ref. 15 and to develop a simple and efficient procedure, which utilizes eigenvalue sensitivity with respect to the follower force. The influence of a tip mass and an elastic foundation on the critical flutter load for a cantilever beam is assessed. Numerical examples show that through the use of the present technique of eigenvalue sensitivity only a few iterations are required to achieve quite accurate results and, therefore, computer time can be considerably reduced.

II. Finite Element Formulation

A uniform cantilever beam on an elastic foundation carrying a tip mass M at its free end $x = L$, at which point it is also subjected to a follower force P, is shown in Fig. 1. To derive the element equations of motion for such an elastic system, the extended Hamilton's principle can be used as follows:

$$\int_{t_1}^{t_2} [\delta(T^e - V^e) + \delta W^e]\, dt = 0 \qquad (1)$$

where T^e and V^e are the kinetic and potential energies of the element, respectively, and δW^e represents the variational work

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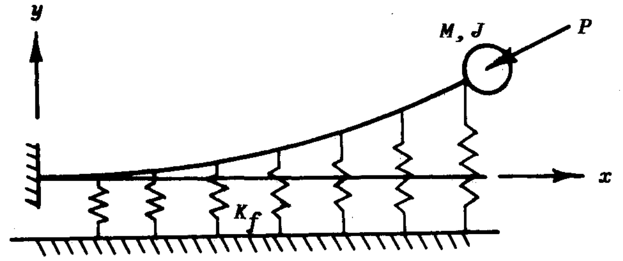

Fig. 1 Cantilever beam with a tip mass resting on an elastic foundation and subjected to a follower force P at $x = L$.

done by nonconservative forces and any forces not accounted for in the potential energy function.

The beam element presented in this work consists of two nodes; each node has the degrees of freedom of lateral displacement w and cross-sectional rotation θ. The potential energy V^e of the beam element of length l including the effects of both shear deformation and elastic foundation is given by

$$V^e = \frac{1}{2}\int_0^l EI(\theta')^2\,dx + \frac{1}{2}\int_0^l kGA(w' - \theta)^2\,dx + \frac{1}{2}\int_0^l K_f w^2\,dx \tag{2}$$

where a prime denotes differentiation with respect to axial distance x, E is the Young's modulus, I the second moment of inertia, k the shear coefficient, G the shear modulus, A the cross-sectional area, and K_f the Winkler foundation modulus.

The kinetic energy T^e of the beam element considering rotatory inertia is given by

$$T^e = \frac{1}{2}\int_0^l \rho A(\dot{w})^2\,dx + \frac{1}{2}\int_0^l \rho I(\dot{\theta})^2\,dx + \frac{1}{2}M\dot{w}^2|_{x=L} + \frac{1}{2}J\dot{\theta}^2|_{x=L} \tag{3}$$

in which a superscript dot indicates differentiation with respect to time t, ρ is the mass density of the beam material, and J is the rotatory inertia of the tip mass. The last two terms in Eq. (3) are included only when the element is connected with the tip mass.

The only variational work included in this study is due to the follower force P. According to the definition of Green strain and under the assumptions that the cantilever beam is inextensional and the follower force follows the normal of the section, the variational work can be expressed as

$$\delta W^e = \int_0^l Pw'(\delta w')\,dx - P[\theta\delta(w)]|_{x=L} \tag{4}$$

where the last term is included only when the element is connected with the tip mass.

In the finite element method, the continuous displacements may be approximated in terms of discretized nodal displacements. The lateral displacement w and cross-sectional rotation θ of a typical point within the element can be related to the nodal displacement vector $\{q^e\}$ as well as the translational and rotational shape function matrices $[N_t]$ and $[N_r]$, respectively, as,

$$w = [N_{t1} \quad N_{t2} \quad N_{t3} \quad N_{t4}]\{q^e\} = [N_t]\{q^e\} \tag{5}$$

$$\theta = [N_{r1} \quad N_{r2} \quad N_{r3} \quad N_{r4}]\{q^e\} = [N_r]\{q^e\} \tag{6}$$

where $\{q^e\} = \{w_1, \theta_1, w_2, \theta_2\}^T$, and

$$N_{t1} = [1 - 3\xi^2 + 2\xi^3 + (1 - \xi)\phi]/(1 + \phi)$$

$$N_{t2} = l[\xi - 2\xi^2 + \xi^3 + (\xi - \xi^2)\phi/2]/(1 + \phi)$$

$$N_{t3} = (3\xi^2 - 2\xi^3 + \xi\phi)/(1 + \phi)$$

$$N_{t4} = l[-\xi^2 + \xi^3 - (\xi - \xi^2)\phi/2]/(1 + \phi)$$

$$N_{r1} = 6(-\xi + \xi^2)/[l(1 + \phi)]$$

$$N_{r2} = [1 - 4\xi + 3\xi^2 + (1 - \xi)\phi]/(1 + \phi)$$

$$N_{r3} = 6(\xi - \xi^2)/[l(1 + \phi)]$$

$$N_{r4} = (-2\xi + 3\xi^2 + \xi\phi)/(1 + \phi)$$

$$\xi = x/l, \qquad \phi = 12EI/(kGAl^2)$$

The shape functions just listed can be evaluated by using the expression of Timoshenko beam static deflection.[16]

The shear strain γ within the element can be obtained by the use of Eqs. (5) and (6). It is

$$\gamma = \frac{dw}{dx} - \theta = ([N_t'] - [N_r])\{q^e\} = [B_s]\{q^e\} \tag{7}$$

With the aid of Eqs. (5–7), the kinetic energy T^e, potential energy V^e, and the variational work δW^e can be written in terms of the nodal displacement vector as,

$$T^e = \tfrac{1}{2}\{\dot{q}^e\}^T[M_t^e]\{\dot{q}^e\} + \tfrac{1}{2}\{\dot{q}^e\}^T[M_r^e]\{\dot{q}^e\} + \tfrac{1}{2}\{\dot{q}^e\}^T[M_d]\{\dot{q}^e\} \tag{8}$$

$$V^e = \tfrac{1}{2}\{q^e\}^T[K_b^e]\{q^e\} + \tfrac{1}{2}\{q^e\}^T[K_s^e]\{q^e\} + \tfrac{1}{2}\{q^e\}^T[K_f^e]\{q^e\} \tag{9}$$

$$\delta W^e = \delta\{q^e\}^T(P[G_C^e])\{q^e\} - \delta\{q^e\}^T(P[G_N])\{q^e\} \tag{10}$$

where

$$[M_t^e] = \int_0^l [N_t]^T\rho A[N_t]\,dx$$

$$[M_r^e] = \int_0^l [N_r]^T\rho I[N_r]\,dx$$

$$[K_b^e] = \int_0^l [N_r']^T EI[N_r']\,dx$$

$$[K_s^e] = \int_0^l [B_s]^T kGA[B_s]\,dx$$

$$[K_f^e] = \int_0^l K_f[N_t]^T[N_t]\,dx$$

$$[G_C^e] = \int_0^l [N_t']^T[N_t']\,dx$$

$[M_d]$: all elements $= 0$, except $M_d(3,3) = M$, $M_d(4,4) = J$;
$[G_N]$: all elements $= 0$, except $G_N(3,4) = 1$.

It should be noted that the present formulation corresponds to the conventional Euler-Bernoulli beam model if the rotatory inertia matrix $[M_r^e]$ and the shear deformation parameter ϕ in the shape functions are all omitted it corresponds to the Rayleigh beam model if only the parameter ϕ is omitted, and if only the rotatory inertia matrix $[M_r^e]$ is omitted, then it is indicated as "shear deformation effect only" in the subsequent study.

Upon substituting Eqs. (8–10) into the extended Hamilton's principle, Eq. (1), and assembling the contribution of each element, the global finite element equation is then obtained as

$$[M]\{\ddot{q}\} + ([K] - P[S])\{q\} = \{0\} \tag{11}$$

where

$$[M] = \sum_e ([M_t^e] + [M_r^e]) + [M_d]$$

$$[K] = \sum_e ([K_b^e] + [K_s^e] + [K_f^e])$$

$$[S] = \sum_e ([G_C^e]) - [G_N]$$

$$\{q\} = \sum_e \{q^e\}$$

All of the matrices just listed are symmetric with the exception of the matrix $[S]$. This makes the governing equation, Eq. (11), of the system non-self-adjoint and reflects the characteristic of nonconservative systems.

III. Stability Analysis and Eigenvalue Sensitivity

A solution of the governing equation, Eq. (11), is now sought in the form

$$\{q\} = \{y\}e^{i\omega t} \tag{12}$$

where ω is a characteristic frequency to be determined. Upon substituting Eq. (12) into Eq. (11), one obtains the eigenvalue problem as

$$([K] - P[S] - \omega^2[M])\{y\} = \{0\} \tag{13}$$

Dealing with the nonconservative problems, it should be noted that instability may occur either dynamically (flutter) or statically (divergence). Divergence occurs when

$$\omega_n = 0, \qquad n = 1, 2, 3, \ldots \tag{14}$$

and flutter occurs when

$$\omega_m = \omega_n \qquad (m \neq n), m, n = 1, 2, 3, \ldots \tag{15}$$

The lowest positive value of the divergence and flutter loads is the critical load and is denoted by P_{cr}.

Corresponding to each eigenvalue ω_i^2 in Eq. (13), there exists a right eigenvector $\{y_i\}$ and a left eigenvector $\{z_i\}$. They are defined by

$$([K] - P[S] - \omega_i^2[M])\{y_i\} = \{0\} \tag{16}$$

$$\{z_i\}^T([K] - P[S] - \omega_i^2[M]) = \{0\}^T \tag{17}$$

The eigenvectors $\{y_i\}$ and $\{z_i\}$ satisfy the biorthogonality relations[17]

$$\{z_i\}^T[M]\{y_j\} = R_i\delta_{ij} \tag{18}$$

$$\{z_i\}^T([K] - P[S])\{y_j\} = \omega_i^2 R_i \delta_{ij} \tag{19}$$

where R_i is the system state modal norm and δ_{ij} is the Kronecker delta function.

The influence of the follower force on the system eigenvalues can be assessed through the eigenvalue sensitivity analysis. The basic idea is to expand the eigenvalues and eigenvectors in a Taylor series about a reference value of the follower force

$$\omega^2 = \omega_0^2 + \left.\frac{\partial(\omega^2)}{\partial P}\right|_0 (\Delta P) \tag{20}$$

where the subscript 0 refers to the evaluation of the term at the reference value.

Premultiplication of Eq. (16) by $\{z_i\}^T$ results in the scalar equation

$$\{z_i\}^T([K] - P[S] - \omega_i^2[M])\{y_i\} = 0 \tag{21}$$

Differentiation of Eq. (21) with respect to the follower force P and making use of the biorthogonality relationships, Eqs. (18) and (19), the eigenvalue sensitivity coefficient due to the follower force is obtained:

$$\frac{\partial(\omega_i^2)}{\partial P} = \frac{\{z_i\}^T[S]\{y_i\}}{R_i} \tag{22}$$

IV. Algorithm

As indicated in Eq. (15), the onset of instability of flutter type occurs when the two lowest natural frequencies coalesce. Suppose that the two lowest natural frequencies, denoted by ω_{1*} and ω_{2*}, are determined from Eq. (13) at a given reference value of the follower force P^*, then a Taylor expansion of the two natural frequencies about P^* yields

$$\omega_1^2 = (\omega_1^2)_* + \left.\frac{\partial(\omega_1^2)}{\partial P}\right|_{P^*} (\Delta P) \tag{23}$$

$$\omega_2^2 = (\omega_2^2)_* + \left.\frac{\partial(\omega_2^2)}{\partial P}\right|_{P^*} (\Delta P) \tag{24}$$

Assuming $\omega_1 = \omega_2$, then ΔP can be determined from Eqs. (23) and (24) as

$$\Delta P = [(\omega_1^2)_* - (\omega_2^2)_*] \bigg/ \left.\frac{\partial(\omega_2^2 - \omega_1^2)}{\partial P}\right|_{P^*} \tag{25}$$

or expressed in another form by the use of Eq. (22)

$$\Delta P = [(\omega_1^2)_* - (\omega_2^2)_*] \bigg/ \left(\frac{\{z_2\}^T[S]\{y_2\}}{R_2} - \frac{\{z_1\}^T[S]\{y_1\}}{R_1}\right) \tag{26}$$

where ΔP is the incremental load and can be regarded as the difference between the reference load P^* and the exact critical flutter load P_{cr}.

After ΔP is determined from Eq. (26), the approximation to the exact critical flutter load is then tried by

$$P_{cr}^* = P^* + \Delta P \tag{27}$$

Now, the new estimated critical flutter load P_{cr}^*, Eq. (27), is substituted into Eq. (13). If there exists at least one pair of the natural frequencies that are complex conjugate, then flutter has occurred. Therefore, the exact critical flutter load P_{cr} is surely bounded in the region

$$P^* < P_{cr} < P_{cr}^* \tag{28}$$

and then the next approximation to the exact critical flutter load is tried by

$$P_{cr}^{**} = (P^* + P_{cr}^*)/2 \tag{29}$$

It is clear that these processes can be repeated for the search of the exact critical flutter load P_{cr} until the difference of the two lowest natural frequencies, ω_1 and ω_2, is less than some acceptable value, or until the incremental load ΔP becomes a very small acceptable value. Typically, this procedure converges in a few iterations.

Table 1 Convergence history of the result of Eq. (30)

Iteration number	Estimated value	Does flutter occur? [checked by Eq. (13)]	Determine ΔP [using Eq. (26), if necessary]	New estimated value in next iteration [using Eq. (27) or (29)]
1	0.0	No	8.7998	$P_{cr}^{*} = 8.7998$
2	8.7998	Yes	—	$P_{cr}^{**} = 4.3999$
3	4.3999	No	1.3756	$P_{cr}^{*} = 5.7755$
4	5.7755	Yes	—	$P_{cr}^{**} = 5.0877$
5	5.0877	No	3.4841×10^{-2}	$P_{cr}^{*} = 5.1225$
6	5.1225	Yes	—	$P_{cr}^{**} = 5.1051$
7	5.1051	No	2.2307×10^{-5}	$P_{cr}^{*} = 5.1051$

Final results: $\omega_1 = 2.4862$, $\omega_2 = 2.4909$, $P_{cr} = 5.1051$

V. Numerical Results and Discussion

In the following computations, the beam is modeled as an assembly of eight finite elements of equal length and the shear coefficient k is taken as 0.85. Also, the following nondimensional quantities are defined:

$$\beta = \bar{K}_f L^4/EI, \quad \mu = M/\rho AL, \quad \eta = J/\rho AL^3, \quad R = (I/A)^{1/2}/L$$

To evaluate the accuracy of the present finite element model and show the convergence rate of the iterative procedure developed here, a numerical example, which was studied by Kounadis,[8] is examined first. In this example, $\mu = \eta = 0.5$, $\beta = 0.0$, and the beam is modeled as the Euler-Bernoulli beam theory. The critical flutter load calculated for this case is

$$P_{cr} = 5.1051(EI/L^2) \tag{30}$$

and the two lowest natural frequencies, ω_1 and ω_2, are

$$\omega_1 = 2.4862(EI/\rho AL^4)^{1/2} \tag{31}$$

$$\omega_2 = 2.4909(EI/\rho AL^4)^{1/2} \tag{32}$$

Comparing the present result with Kounadis' solution $P_{cr} = 5.11$, the accuracy of the present finite element model is demonstrated. The convergence history of the calculation of Eq. (30) is listed in Table 1. The iterative procedure is initiated by trying an arbitrarily chosen value of zero as the beginning trial load of the exact critical flutter load. Following the procedures described in a previous section, the final result is obtained after only seven iterations under the criterion that the two lowest natural frequencies have a difference <0.5% compared to each other. Because the iterations required in the conventional trial-and-error technique depend on both the initial guess of the critical flutter load and the load increment used in each iteration, the final result is usually obtained after many iterations. In contrast, an arbitrarily chosen initial guess can be used in the present eigenvalue sensitivity technique, thus reducing the computational time if the same finite element equations are employed.

The effects of transverse shear deformation and rotatory inertia on the critical flutter load P_{cr} of a cantilever beam ($\mu = \eta = 0$) with the slenderness parameter $R = 0.05$ and resting on an elastic foundation is investigated next. Four different beam models are used and results are compared in Fig. 2. Upon investigation of Fig. 2, it is seen that besides the Beck's column (i.e., the column modeled as the Euler-Bernoulli beam theory), the transverse shear deformation on the critical flutter load is completely independent of the Winkler's foundation modulus parameter β. In fact, the effect of the elastic foundation is only to shift (raise) the flutter frequency (frequency at which $\omega_1 = \omega_2$). It is also observed that transverse shear deformation and rotatory inertia each can decrease the critical flutter load; the critical flutter load is especially affected by the presence of shear deformation.

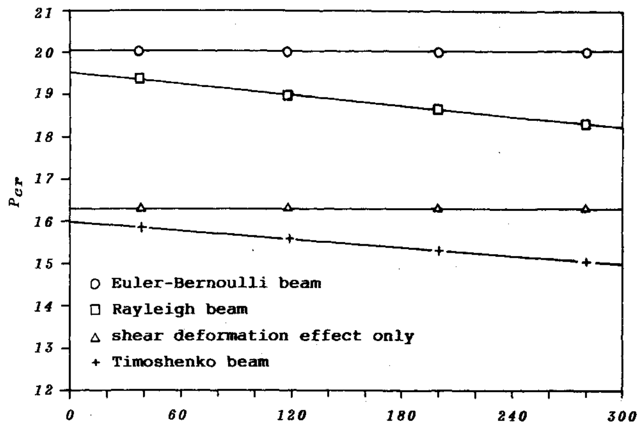

Fig. 2 Variation of the critical flutter load P_{cr} vs the foundation parameter β for $\mu = \eta = 0$, $R = 0.05$.

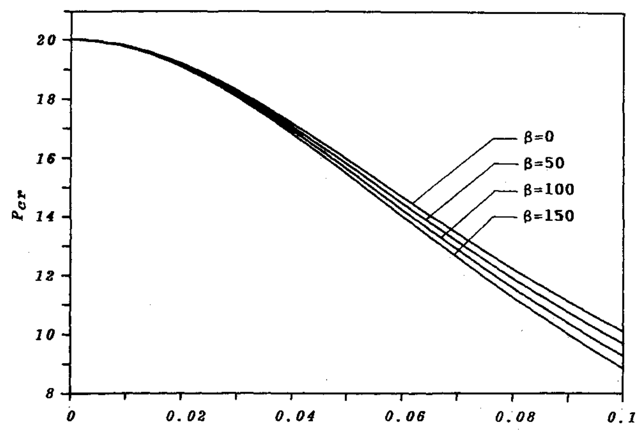

Fig. 3 Dependence of the critical flutter load P_{cr} on the slenderness parameter R for four values of β with $\mu = \eta = 0$.

The dependence of the critical flutter load on the slenderness parameter R varying from 0.001 to 0.1 for different values of the foundation parameter β is plotted in Fig. 3. As the slenderness parameter becomes small, the critical flutter load approaches the value of 20.05, which is Beck's result.[3]

The variation of critical flutter load P_{cr} vs the mass ratio μ, over the range $0 < \mu < \infty$, is plotted in Fig. 4 for four values of β with $\eta = 0$ and $R = 0.05$. As is seen in Fig. 4, an increase in β may decrease or increase the value of P_{cr}; however, P_{cr} is decreased only when μ is located in the small value range. The shapes of the curves in Fig. 4 are strongly dependent on the value of the foundation modulus parameter β. To describe this more precisely, two curves are illustrated. For $\beta = 0$, the value of P_{cr} initiates at 16.00 for $\mu = 0$, attains the minimum value at 13.62 for $\mu = 0.45$, and terminates at

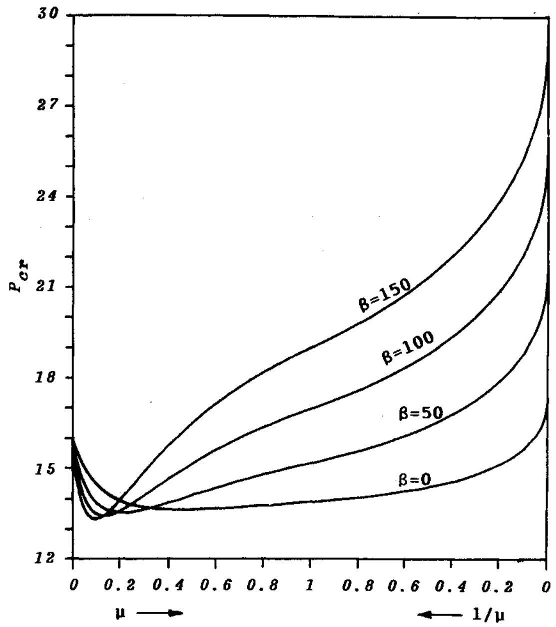

Fig. 4 Variation of the critical flutter load P_{cr} vs the mass ratio μ for four values of β with $\eta = 0$ and $R = 0.05$.

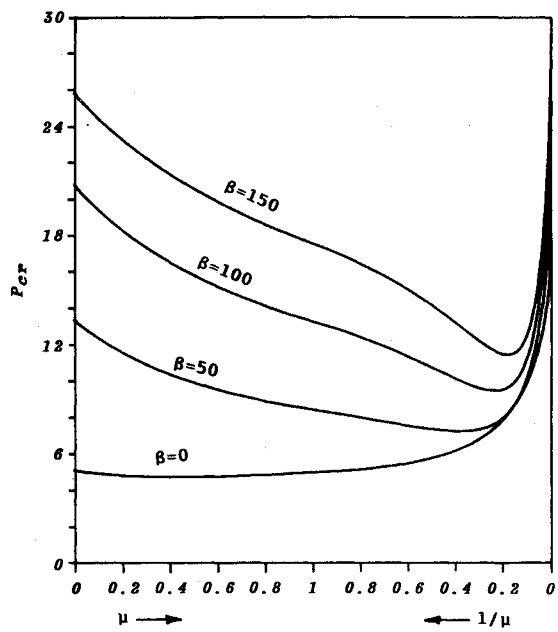

Fig. 5 Variation of the critical flutter load P_{cr} vs the mass ratio μ for four values of β with $\eta = 0.5$ and $R = 0.05$.

17.30 for $1/\mu = 0$. On the other hand, for the curve of $\beta = 150$, P_{cr} initiates at 15.50, attains the minimum value at 13.32 at $\mu = 0.1$, and terminates at 29.15.

Finally, the influence of the rotatory inertia of the tip mass on the value of the critical flutter load is presented in Fig. 5 for $\eta = 0.5$ and $R = 0.05$. Because of the presence of the rotatory inertia of the tip mass, the curves for P_{cr} in Fig. 5 exhibit an entirely opposite trend compared with Fig. 4. For a given value of μ, the critical flutter load is always increased as the foundation modulus parameter β increases.

VI. Conclusions

A finite element model and the eigenvalue sensitivity technique are combined to study the stability of nonconservative systems. Numerical results show that the combined technique can yield a highly accurate solution within only a few iterations. Use of this technique considerably improves the convergence rate and circumvents an arduously computational task.

Numerical examples also show that, in the absence of a tip mass, the elastic foundation does not affect the critical flutter load if the rotatory inertia of the cantilever beam is not taken into account. Its effect is only to raise the natural frequencies.

On the other hand, if the tip mass is considered, the critical flutter load is strongly dependent on the elastic foundation. Moreover, when the effect of rotatory inertia of the tip mass is retained, for a given value of the mass ratio μ, the critical flutter load always increases as the foundation modulus is increased.

References

[1]Bolotin, V. V., *Nonconservative Problems of the Theory of Elastic Stability*, Pergamon, New York, 1963.

[2]Leipholz, H., *Stability of Elastic Systems*, Sijthoff & Noordhoff, Amsterdam, The Netherlands, 1980.

[3]Beck, M., "Die Knicklast des Einseitig Eingespannten Tangential Gedrückten Stabes," *Zeitschrift für Angewandte Mathematik und Physik*, Vol. 3, 1952, pp. 225–228.

[4]Pflüger, A., "Zur Stabilitat des Tangential Gedrückten Stabes," *Zeitschrift für Angewandte Mathematik und Mechanik*, Vol. 35, 1955, p. 191.

[5]Nemat-Nasser, S., "Instability of a Cantilever Under a Follower Force According to Timoshenko Beam Theory," *Journal of Applied Mechanics*, Vol. 34, No. 2, 1967, pp. 484–485.

[6]Sundararajan, C., "Influence of an Elastic End Support on the Vibration and Stability of Beck's Column," *International Journal of Mechanical Sciences*, Vol. 18, No. 5, 1976, pp. 239–241.

[7]Kounadis, A. N., and Katsikadelis, J. T., "Shear and Rotatory Inertia Effect on Beck's Column," *Journal of Sound and Vibration*, Vol. 49, No. 2, 1976, pp. 171–178.

[8]Kounadis, A. N., "Stability of Elastically Restrained Timoshenko Cantilevers with Attached Masses Subjected to a Follower Force," *Journal of Applied Mechanics*, Vol. 44, No. 4, 1977, pp. 731–736.

[9]Katsikadelis, J. T., and Kounadis, A. N., "Flutter Loads of a Timoshenko Beam-Column Under a Follower Force Governed by Two Variants of Equations of Motion," *Acta Mechanica*, Vol. 48, No. 4, 1983, pp. 209–217.

[10]Anderson, G. L., "The Influence of Rotatory Inertia, Tip Mass, and Damping on the Stability of a Cantilever Beam on an Elastic Foundation," *Journal of Sound and Vibration*, Vol. 43, No. 3, 1975, pp. 543–552.

[11]Barsoum, R. S., "Finite Element Method Applied to the Problem of Stability of a Nonconservative System," *International Journal for Numerical Methods in Engineering*, Vol. 3, No. 1, 1971, pp. 63–87.

[12]Sundararamaiah, V., and Rao, G. V., "Effect of Shear Deformation and Rotatory Inertia on the Stability of Beck's and Leipholz's Column," *AIAA Journal*, Vol. 18, No. 1, 1980, pp. 124–125.

[13]Park, Y. P., "Dynamic Stability of a Free Timoshenko Beam Under a Controlled Follower Force," *Journal of Sound and Vibration*, Vol. 113, No. 3, 1987, pp. 407–415.

[14]Chen, L. W., and Yang, J. Y., "Nonconservative Stability of a Bimodulus Beam Subjected to a Follower Force," *Computers & Structures*, Vol. 32, No. 5, 1989, pp. 987–995.

[15]Pedersen, P., and Seyranian, A. P., "Sensitivity Analysis for Problems of Dynamic Stability," *International Journal of Solids and Structures*, Vol. 19, No. 4, 1983, pp. 315–335.

[16]Archer, J. S., "Consistent Matrix Formulations for Structural Analysis Using Finite-Element Techniques," *AIAA Journal*, Vol. 3, No. 10, 1965, pp. 1910–1918.

[17]Inman, D. J., *Vibration with Control Measurement and Stability*, Prentice-Hall, Englewood Cliffs, NJ, 1989, p. 56.